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Analytic aspects of automorphic forms

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This course will be a survey and discussion of the L^2 spectral theory of automorphic forms, in simple cases, as well as higher-rank. The operators involved in the decomposition include the Casimir operator, in the universal enveloping algebra, and its descendant the invariant Laplacian on G/K and $\Gamma\backslash G/K$. For higher-rank groups, the center of the universal enveloping algebra, made explicit by Harish-Chandra's isomorphism, is larger. We do also care about spherical Hecke operators, whose modern incarnation is as integral operators on the non-archimedean factors of the adele group.

With Gelfand's definition of *cuspform*, that all constant terms along unipotent radicals of all rational parabolics are identically zero, the basic decomposition is that $L^2(\Gamma\backslash G/K)$ is an orthogonal sum of the space of L^2 cuspforms, and the space spanned by *pseudo-Eisenstein series*. And the space of L^2 cuspforms decomposes *discretely*, that is, has an orthonormal basis consisting of eigenfunctions for the Laplacian. In fact, the proof shows that that there is a *compact* self-adjoint operator on L^2 cuspforms commuting with the Laplacian. Thus, the eigenspaces are finite-dimensional.

Those finite dimensional eigenspaces are stable under the (self-adjoint) spherical Hecke operators, which commute with the Laplacian, so there is a further decomposition.

The spaces spanned by pseudo-Eisenstein series have *continuous* decompositions, in terms of non-L^2 (genuine) Eisenstein series, attached to cuspidal data on Levi components of parabolics.

There is a corresponding Plancherel theorem for this decomposition of $L^2(\Gamma\backslash G/K)$, in terms of the spectral coefficients.

That Plancherel theorem allows a good characterization of *global automorphic Sobolev spaces*. This allows rigorous discussion of automorphic Green's functions, as solutions to distributional differential equations $(\Delta-\lambda)u=\delta$, for example. It also facilitates understanding and rigorization of Colin de Verdiere's speculations on connections between unbounded self-adjoint operators on spaces of automorphic forms, and the Riemann Hypothesis.
